Section 5.1

2 If a 3 by 3 matrix has $\det A = \frac{1}{2}$, find $\det(2A)$ and $\det(-A)$ and $\det A^2$ and $\det(A^{-1})$.

Solution: $\det(2A) = 4$, $\det(-A) = -\frac{1}{2}$, $\det A^2 = \frac{1}{4}$, and $\det(A^{-1}) = 2$.

3 True or false, with a reason if true or a counterexample if false:

(a) The determinant of $I + A$ is $1 + \det A$.

False, example with $A = I$ being the two by two identity matrix. Then $\det(I+A) = \det(2I) = 4$ and $1 + \det A = 2$.

(b) The determinant of ABC is $|A||B||C|$.

True, the determinant of a product is the product of the determinants.

(c) The determinant of $4A$ is $4|A|$.

False, the determinant of $4A$ is $4^n|A|$ if A is an n by n matrix.

(d) The determinant of $AB - BA$ is zero.

False, example $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$AB - BA = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} - \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix}$$

with determinant bc. This is not zero unless $b = 0$ or $c = 0$.

4 Which row exchanges show that these "reverse identity matrices" J_3 and J_4 have $|J_3| = -1$ but $|J_4| = +1$?

$$\det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = -1 \qquad \text{but} \qquad \det \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = +1.$$

In the first case exchanging the first and third rows makes J_3 the identity matrix. In the second case exchanging the first and fourth, and then the second and third row makes J_4 the identity. An even number of row changes (for J_4) does not change the determinant, an odd number (for J_3) changes the sign. Since the determinant of the identity matrix is 1 in any dimension, this shows the claim.

12 The inverse of a 2 by 2 matrix seems to have determinant = 1:

$$\det A^{-1} = \det \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{ad - bc}{ad - bc} = 1.$$

What is wrong with this calculation? What is the correct $\det A^{-1}$?

Multiplying an n by n matrix by a factor multiplies the determinant by the n-th power of this factor, in this case by $\frac{1}{(ad-bc)^2}$, not $\frac{1}{ad-bc}$. The correct calculation is

$$\det A^{-1} = \det \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{ad - bc}{(ad - bc)^2} = \frac{1}{ad - bc}.$$

28 True or false (give a reason if true or a 2 by 2 example if false):

(a) If A is not invertible then AB is not invertible.

True, $|AB| = |A||B| = 0|B| = 0$.

(b) The determinant of A is always the product of its pivots.

False, there might be an odd number of row exchanges. Example $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has pivots 1 and 1, but determinant -1.

(c) The determinant of $A - B$ equals det A - det B.

False, example $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ both have determinant 0, but $A - B = I$ has determinant 1.

Section 5.2

3 Show that $\det A = 0$, regardless of the five nonzeros marked by x's:

$$A = \begin{bmatrix} x & x & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}.$$

What are the cofactors of row 1? What is the rank of A? What are the 6 terms in $\det A$.

The first two columns of A are both multiples of $(1, 0, 0)$, so they are dependent, A is not invertible, and so $\det A = 0$. The cofactors of row 1 are

$$C_{11} = \begin{vmatrix} 0 & x \\ 0 & x \end{vmatrix} = 0, \quad C_{12} = -\begin{vmatrix} 0 & x \\ 0 & x \end{vmatrix} = 0, \quad C_{13} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0.$$

As seen above, the first and second column are dependent. The first and third column are independent, though, so the rank is 2. The 6 terms in $\det A$ are all zero.

5 Place the smallest number of zeros in a 4 by 4 matrix that will guarantee $\det A = 0$. Place as many zeros as possible while still allowing $\det A \neq 0$.

Placing 4 zeros in the first row (or any other row or column) will guarantee $\det A = 0$. Placing zeros in all off-diagonal entries will still allow $\det A \neq 0$ (if all the diagonal entries are non-zero).

23 With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A||D| \qquad \text{but} \qquad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |C||B|.$$

(a) Why is the first statement true? Somehow B doesn't enter.

Cofactor expansion along the first column gives

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & b_{21} & b_{22} \\ 0 & d_{11} & d_{12} \\ 0 & d_{21} & d_{22} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & b_{11} & b_{12} \\ 0 & d_{11} & d_{12} \\ 0 & d_{21} & d_{22} \end{vmatrix} = a_{11}a_{22} \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix} - a_{21}a_{12} \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}$$

$$= (a_{11}a_{22} - a_{21}a_{12}) \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix} = |A||D|.$$

(b) Show by example that equality fails (as shown) when C enters.

For $A = D = 0$ and $B = C = I$ we have $|A||D| - |B||C| = 0 - 1 = -1$, but

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} = 1.$$

(No need for calculations here, this matrix can be transformed into the identity matrix by two row exchanges, so the determinant is the same as the identity matrix.)

Section 6.1

2 Find the eigenvalues and eigenvectors of these two matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

The eigenvalues of A are -1 and 5, those of $A+I$ are 0 and 6. The corresponding eigenvectors are $(-2, 1)$ and $(1, 1)$.

$A + I$ has the <u>same</u> eigenvectors as A. Its eigenvalues are <u>greater</u> by 1.

5 Find the eigenvalues of A and B (easy for triangular matrices) and $A + B$:

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A + B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

Eigenvalues of both A and B are 3 and 1, eigenvalues of $A + B$ are 3 and 5.

Eigenvalues of $A + B$ *are not equal to* eigenvalues of A plus eigenvalues of B.

21 The eigenvalues of A equal the eigenvalues of A^T. This is because $\det(A - \lambda I)$ equals $\det(A^T - \lambda I)$. That is true because <u>$(A - \lambda I)^T = A^T - \lambda I$ and the determinant does not change under transposition of matrices</u>. Show by an example that the eigenvectors of A and A^T are *not* the same.

Example: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has only 0 as an eigenvalue, with eigenvector $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. However, $A^T\mathbf{x} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not a multiple of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not an eigenvector of A^T.

29 (Review) Find the eigenvalues of A, B, and C:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

Eigenvalues of A are 1, 4, and 6, eigenvalues of B are 2 and $\pm\sqrt{3}$, eigenvalues of C are 0, 0, and 6.

Section 6.2

2 If A has $\lambda_1 = 2$ with eigenvector $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda_2 = 5$ with $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, use $S\Lambda S^{-1}$ to find A. No other matrix has the same λ's and x's.

$$A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}.$$

11 True or false: If the eigenvalues of A are 2, 2, 5 then the matrix is certainly

(a) invertible

True, because the eigenvalues are non-zero.

(b) diagonalizable

False, it might not be because of the repeated eigenvalue 2.

(c) not diagonalizable

False, it might be diagonalizable, e.g., it could just be the diagonal matrix with diagonal entries 2, 2, and 5.

14 The matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalizable because the rank of $A - 3I$ is <u>one</u>. Change one entry to make A diagonalizable. Which entries could you change?

Changing the 1 to a 0 obviously makes the matrix diagonal, and thus diagonalizable. Changing any of the 3's to a different number makes it diagonalizable because it will have two different eigenvalues. Changing the 0 to something else will also lead to two different eigenvalues and make the matrix diagonalizable.

16 (Recommended) Find Λ and S to diagonalize $A_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix}$. What is the limit of Λ^k as $k \to \infty$? What is the limit of $S\Lambda^k S^{-1}$? In the columns of the limiting matrix you see the <u>eigenvectors to $\lambda = 1$</u>.

Eigenvalues are 1 and $-.3$, so $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -.3 \end{bmatrix}$. Corresponding eigenvectors are $\begin{bmatrix} .9 \\ .4 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, so $S = \begin{bmatrix} .9 & -1 \\ .4 & 1 \end{bmatrix}$. The limit of Λ^k is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, the limit of $A^k = S\Lambda^k S^{-1}$ is $\frac{1}{1.3} \begin{bmatrix} .9 & .9 \\ .4 & .4 \end{bmatrix}$.