Cauchy's Integral Formula and Power and Laurent Series Expansion

Cauchy Integral Formula. Goursat's technique removed the additional condition of the continuity of the derivative of the holomorphic function from Cauchy's theorem. Actually derivatives of all orders of a holomorphic function are always continuous. This is proved by using Cauchy's integral formula for holomorphic functions which is a consequence of applying the theorem of Cauchy-Goursat to the difference of a holomorphic function at some prechosen point. The precise statement of Cauchy's integral formula is as follows.

Statement of Cauchy's Integral Formula. Let f be a holomorphic function on an open subset of $\mathbb{C}$ which contains the counter-clockwise simple closed piecewise smooth curve C and all points of $\mathbb{C}$ enclosed by C. Then for every point a inside the domain enclosed by C the value $f(a)$ at a can be expressed in terms of the values of $f(z)$ on C by

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz.$$

Proof of Cauchy's Integral Formula. Let Ω be the domain enclosed by C and let C_r be $\{|z-a|=r\}$ for $r>0$ sufficiently small so that

$$\overline{D_r(a)} := \{|z-a|\leq r\}$$

is contained in Ω.

The key of the proof is to apply the theorem of Cauchy-Goursat to the difference quotient

$$\frac{f(z)-f(a)}{z-a}$$

on the domain

$$\Omega_r = \Omega - \overline{D_r(a)}$$

whose boundary consists of two parts, one of which is C in the counterclockwise sense, with the other being C_r in the clockwise sense. By the theorem of Cauchy-Goursat,

$$\int_C \frac{f(z)-f(a)}{z-a}\,dz = \int_{C_r} \frac{f(z)-f(a)}{z-a}\,dz.$$

As $r \to 0$, the right-hand side can be estimated to approach 0 as follows.

$$\left|\int_{C_r} \frac{f(z)-f(a)}{z-a} dz\right| \leq 2\pi r \sup_{z\in C_r}\left|\frac{f(z)-f(a)}{z-a}\right| \to 0 \quad \text{as} \quad r \to 0.$$

Thus,

$$\int_C \frac{f(z)-f(a)}{z-a}\, dz = 0 \quad \text{and} \quad \int_C \frac{f(z)}{z-a}\, dz = \int_C \frac{f(a)}{z-a}\, dz.$$

We now compute the right-hand side as follows.

$$\begin{aligned}\int_C \frac{f(a)}{z-a}\, dz &= f(a)\int_C \frac{1}{z-a}\, dz \\ &= f(a)\int_{C_r} \frac{1}{z-a}\, dz \quad \left(\text{Cauchy theorem applied to } \frac{1}{z-a} \text{ on } \Omega_r\right) \\ &= f(a)\int_{\theta=0}^{2\pi} \frac{d(a+re^{i\theta})}{(a+re^{i\theta})-a} \quad (\text{computed by angle parametrization of } C_r) \\ &= f(a)\int_{\theta=0}^{2\pi} \frac{ire^{i\theta}d\theta}{(re^{i\theta})} = 2\pi i f(a).\end{aligned}$$

Q.E.D.

Remark. Cauchy's integral formula holds in the following more general situation. Let Ω be a bounded domain (*i.e.*, connected open subset) in $\mathbb{C}$ with piecewise smooth boundary. Let C be the boundary $\partial\Omega$ of Ω. It is possible that the boundary C consists of a finite number of disjoint components. Let U be an open neighborhood of the topological closure $\bar{\Omega}$ of Ω in $\mathbb{C}$. For any holomorphic function $f(z)$ on U, Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a} dz$$

holds for $a \in \Omega$.

As in the proof of Goursat's theorem for the general case, we can choose a finite number of line segments $\gamma_1, \cdots, \gamma_p$ in $\Omega - \{a\}$ transversal to $\partial\Omega$ with end-points in the smooth part of $\partial\Omega$ such that $\Omega - \cup_{j=1}^p \gamma_j$ is simply connected. Then we apply Cauchy's integral formula or Cauchy's theorem to each component of $\Omega - \cup_{j=1}^p \gamma_j$. Each integral over each artificially introduced line segment γ_j occurs twice with opposite orientations of γ_j, resulting in the cancellations of the integrals over such artificially introduced line segments.

Cauchy Kernel. For a holomorphic function $f(z)$ on some open neighborhood of the topological closure of a bounded open subset Ω of $\mathbb{C}$ with piecewise smooth boundary, the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(\zeta)d\zeta}{\zeta - z}$$

for $z \in \Omega$ tells us that the function $f(z)$ as a function of z for $z \in \Omega$ is simply a "continuous linear combination" of the function

$$\frac{1}{\zeta - z}$$

parametrized by $\zeta \in \partial\Omega$. Here the "continuous linear combination" is given by an integral with respect to $d\zeta$ over $\zeta \in \partial\Omega$ with the appropriate "weight" $f(\zeta)$. The "weight" $f(\zeta)$ plays the rôle of the coefficients in a linear combination in the form of a discrete sum. The function

$$\frac{1}{\zeta - z}$$

of the complex variable z with ζ regarded as a parameter is known as the *Cauchy kernel*. In a way the *Cauchy kernel*, like the kernel of a plant seed, generates all holomorphic functions by using "continuous linear combinations" which are integrals with respect to the parameter variable.

Power Series Expansion and Cauchy's integral formula for derivative of holomorphic function. Suppose $f(z)$ is a holomorphic function on open disk $\{|z-a| < R\}$ (with $a \in \mathbb{C}$ and $R > 0$). For fixed $z \in \mathbb{C}$ and $0 < r < R$ with $|z-a| < r$ and for variable $\zeta \in \mathbb{C}$ satisfying $|\zeta - a| = r$, we can use the geometric series expansion of the Cauchy kernel

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a - (z-a)} = \frac{1}{\zeta - a}\frac{1}{1 - \frac{z-a}{\zeta-a}} = \sum_{n=0}^{\infty}\frac{(z-a)^n}{(\zeta-a)^{n+1}}$$

to express $f(z)$ as a power series on the open disk $\{|z-a| < r\}$ with center a.

With z fixed and with ζ as the variable, the above geometric series converges uniformly on $|\zeta - a| = r$. Term-by-term integration after multiplication by $f(\zeta)d\zeta$ over $\{|\zeta - a| = r\}$ yields

$$(*) \qquad \int_{|\zeta-a|=r}\frac{f(\zeta)d\zeta}{\zeta - z} = \sum_{n=0}^{\infty}\left(\int_{|\zeta-a|=r}\frac{f(\zeta)d\zeta}{(\zeta-a)^{n+1}}\right)(z-a)^n.$$

We have the following theorem on power series expansion.

Theorem on Power Series Expansion. For $a \in \mathbb{C}$ and $R > 0$ the power series expansion of a holomorphic function $f(z)$ holomorphic on $\{|z-a| < R\}$

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

holds for $|z-a| < R$, where

$$c_n = \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta) d\zeta}{(\zeta - a)^{n+1}}$$

for any $0 < r < R$.

Proof of Theorem on Power Series Expansion. First, we observe that as a result of the theorem of Cauchy-Goursat, the coefficient c_n is independent of the choice $0 < r < R$, because for $0 < r < \hat{r} < R$, the application of the theorem of Cauchy-Goursat to the holomorphic function

$$\frac{f(\zeta)}{(\zeta - a)^{n+1}}$$

on the annulus domain $\{r < |z-a| < \hat{r}\}$ yields

$$\frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta) d\zeta}{(\zeta - a)^{n+1}} = \frac{1}{2\pi i} \int_{|\zeta - a| = \hat{r}} \frac{f(\zeta) d\zeta}{(\zeta - a)^{n+1}}.$$

To verify that the power series expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

holds for $|z-a| < R$, we need only choose $0 < r < R$ with $|z-a| < r$ and observe that the left-hand side of $(*)$ above is equal to $f(z)$ by Cauchy's formula for holomorphic functions. Q.E.D.

Since a power series can be differentiated term-by-term within its open disk of convergence, when we differentiate k times the power series expansion of a holomorphic function on $\{|z-a| < r\}$ and evaluate at the point $z = a$, we obtain

$$f^{(k)}(z) = k! c_k$$

for $k \in \mathbb{N}$. This means that the following theorem holds.

Theorem on Cauchy's Integral Formula for Derivatives of Holomorphic Functions. If $f(z)$ is a holomorphic function on $\{|z-a|<R\}$ with $a \in \mathbb{C}$ and $R>0$, then for any z in the open disk $\{|z-a|<R\}$,

$$f^{(k)}(z) = \frac{k!}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)d\zeta}{(\zeta-z)^{k+1}}$$

for any r with $|z-a|<r<R$.

Remark. We now know that a holomorphic function $f(z)$ is always complex differentiable to any order (at any point of the domain where it is holomorphic).

Laurent Series Expansion of Holomorphic Function on Annulus. The argument used to get the power series expansion of a holomorphic function on an open disk can be applied to any holomorphic function defined on an open annulus to yield a Laurent series. A Laurent series centered at a means an infinite sum of the form

$$\sum_{n=-\infty}^{\infty} c_n(z-a)^n.$$

It is reduced to a power series centered at a when $c_n=0$ for all $n \leq -1$.

Theorem on Laurent Series Expansion. Let $0 \leq R_1 < R_2 < \infty$. Let $f(z)$ be a holomorphic function on an open annulus $\{R_1 < |z-a| < R_2\}$. Then the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$$

of $f(z)$ holds on the open annulus $R_1 < |z-a| < R_2$, where

$$c_n = \frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)d\zeta}{(\zeta-a)^{n+1}}$$

for $n \in \mathbb{Z}$ and for any $R_1 < r < R_2$. Moreover, the convergence of

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$$

is absolute and uniform on $r_1 \leq |z-a| \leq r_2$ for any $R_1 < r_1 < r_2 < R_2$. Note that the integral defining c_n above is independent of the choice of $r \in (R_1, R_2)$ on account of the theorem of Cauchy-Goursat.

Proof of Theorem on Laurent Series Expansion. Take any z with $R_1 < |z-a| < R_2$. Choose $R_1 < r_1 < r_2 < R_2$ with $r_1 \leq |z-a| \leq r_2$. By Cauchy's theorem

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta-a|=r_2}\frac{f(\zeta)d\zeta}{\zeta-z} - \frac{1}{2\pi i}\int_{|\zeta-a|=r_1}\frac{f(\zeta)d\zeta}{\zeta-z}.$$

For the first integral on the right-hand side we use the preceding method of power series expansion for the kernel and get

$$\frac{1}{\zeta-z} = \sum_{n=0}^{\infty}\frac{(z-a)^n}{(\zeta-a)^{n+1}}$$

for $|z-a| < r_2 = |\zeta-a|$. For the second integral on the right-hand side we exchange the rôles of $z-a$ and $\zeta-a$ when we use the preceding method of power series expansion for the kernel so that we get

$$\frac{1}{\zeta-z} = -\sum_{n=0}^{\infty}\frac{(\zeta-a)^n}{(z-a)^{n+1}}$$

for $|\zeta-a| = r_1 < |z-a|$. After we multiply by $f(\zeta)d\zeta$ and integrate the two power series expansions over $|\zeta-a| = r_2$ and $|\zeta-a| = r_1$ respectively, we get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{|\zeta-a|=r_2}\frac{f(\zeta)d\zeta}{\zeta-z} - \frac{1}{2\pi i}\int_{|\zeta-a|=r_1}\frac{f(\zeta)d\zeta}{\zeta-z} \\ &= \sum_{n=0}^{\infty}(z-a)^n\left(\frac{1}{2\pi i}\int_{|\zeta-a|=r_2}\frac{f(\zeta)d\zeta}{(\zeta-a)^{n+1}}\right) + \sum_{n=0}^{\infty}\frac{1}{(z-a)^{n+1}}\left(\frac{1}{2\pi i}\int_{|\zeta-a|=r_1}(\zeta-a)^n f(\zeta)d\zeta\right) \\ &= \sum_{n=0}^{\infty}(z-a)^n\left(\frac{1}{2\pi i}\int_{|\zeta-a|=r_2}\frac{f(\zeta)d\zeta}{(\zeta-a)^{n+1}}\right) + \sum_{m=-1}^{-\infty}\left(\frac{1}{2\pi i}(z-a)^m\int_{|\zeta-a|=r_1}\frac{f(\zeta)d\zeta}{(\zeta-a)^{m+1}}\right) \end{aligned}$$

where in the last step we change the indexing variable n in the second summation to $-m-1$ so that

$$\frac{1}{(z-a)^{n+1}} = (z-a)^m \quad \text{and} \quad (\zeta-a)^n = \frac{1}{(\zeta-a)^{m+1}}$$

and the range $\{0\} \cap \mathbb{N}$ of the old indexing variable n becomes the range $-\mathbb{N}$ of the new indexing variable m. This finishes the proof of the Laurent series expansion, because

$$c_n = \frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)d\zeta}{(\zeta-a)^{n+1}} = \frac{1}{2\pi i}\int_{|\zeta-a|=r_1}\frac{f(\zeta)d\zeta}{(\zeta-a)^{n+1}} \quad \text{for} \quad n \geq 0$$

and

$$c_n = \frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)d\zeta}{(\zeta-a)^{n+1}} = \frac{1}{2\pi i}\int_{|\zeta-a|=r_2}\frac{f(\zeta)d\zeta}{(\zeta-a)^{n+1}} \quad \text{for} \quad n<0$$

by Cauchy's theorem applied to the holomorphic function $\frac{f(\zeta)}{(\zeta-a)^{n+1}}$ of the complex variable ζ on $R_1<|\zeta-a|<R_2$. Q.E.D.

Uniqueness of Coefficients of Laurent Series. Suppose

$$f(z)=\sum_{n=-\infty}^{\infty}c_n(z-a)^n$$

is absolute and uniform on $r_1\leq|z-a|\leq r_2$ for any $R_1<r_1<r_2<R_2$. Then, for $r_1<r<r_2$ and $k\in\mathbb{Z}$,

$$\begin{aligned}\frac{1}{2\pi i}\int_{|z-a|=r}\frac{f(z)dz}{(z-a)^{k+1}} &= \sum_{n=-\infty}^{\infty}c_n\,\frac{1}{2\pi i}\int_{|z-a|=r}(z-a)^{n-k-1}dz\\ &= \sum_{n=-\infty}^{\infty}c_n\,\frac{1}{2\pi i}\int_{\theta=0}^{2\pi}\left((a+re^{i\theta})-a\right)^{n-k-1}d(a+re^{i\theta})\\ &= \sum_{n=-\infty}^{\infty}c_n\,\frac{1}{2\pi i}\int_{\theta=0}^{2\pi}r^{n-k}ie^{i(n-k)\theta}d\theta\\ &= c_k,\end{aligned}$$

because

$$\int_{\theta=0}^{2\pi}e^{im\theta}d\theta=0 \quad \text{for} \quad m\in\mathbb{Z}-\{0\}$$

and

$$\int_{\theta=0}^{2\pi}d\theta=2\pi.$$

Isolated Singularity of Holomorphic Function and Residue. Let $a\in\mathbb{C}$ and $R>0$. Let $f(z)$ be holomorphic function on the punctured open disk $\{0<|z-a|<R\}$ of radius R centered at a. From the Theorem on Laurent series expansion we know that $f(z)$ admits the Laurent series expansion

$$f(z)=\sum_{n=-\infty}^{\infty}c_n(z-a)^n$$

on the punctured open disk $\{0<|z-a|<R\}$.

Definition. The point a is called an *isolated singularity* of $f(z)$. The part

$$\sum_{n\leq -1} c_n(z-a)^n$$

is known as the *principal part* of the Laurent series of $f(z)$.

When the principal part is 0, the isolated singularity a is called a *removable singularity* of f, in which case the Laurent series is actually a power series and the function $f(z)$ can be extended to a holomorphic function on the open disk $\{|z-a|<R\}$ (without the puncture).

When there are only a finite number of nonzero terms in the non identically zero principal part, the isolated singularity a is called a *pole* of f. The *order* of the pole is the largest positive integer k such that c_{-k} is nonzero so that the principal part is

$$\sum_{-k\leq n\leq -1} c_n(z-a)^n$$

with $c_n \neq 0$ for $n=-k$.

When there are an infinite number of nonzero terms in the principal part, the isolated singularity a is called an *essential singularity* of f.

The coefficient c_n when $n=-1$ is called the *residue* of f at the isolated singularity a and is denoted by $\mathrm{Res}_a f$. Thus,

$$\int_{|z-a|=r} f(z)dz = 2\pi i\,\mathrm{Res}_a f$$

for any $0<r<R$.

Residue at a Pole. If the isolated singularity a is a pole of order $\leq k$ for f, another way to compute the residue of f at a is

$$\mathrm{Res}_a f = \frac{1}{(k-1)!}\,\frac{d^{k-1}}{dz^{k-1}}\left((z-a)^k f(z)\right)\Big|_{z=a},$$

because

$$(z-a)^k f(z) = (z-a)^k \sum_{n=-k}^{\infty} c_n(z-a)^n$$

is a power series whose $(k-1)$-st coefficient can be computed by $(k-1)$-fold differentiation at $z=a$.

Residues play a very important role in the application of complex analysis to the computation of definite integrals.

Remark on Complex-Analytic Analogue of Fundamental Theorem of Calculus. In real analysis the fundamental theorem of calculus for a real-valued function $y = g(x)$ of a real variable is

$$g(x) = g(a) + \int_{t=a}^{x} g'(t)dt$$

which expresses $g(x)$ in terms of its value at some initial point a and the integral of its derivative. Cauchy's integral formula is not the complex-analytic analogue of the fundamental theorem of calculus. Though we will not go into the complex-analytic analogue at this point, we would like to state it here. Let Ω be a bounded domain in $\mathbb{C}$ with piecewise smooth boundary and let $f(z)$ be a *smooth* function on an open neighborhood U of the topological closure $\bar{\Omega}$ of Ω in $\mathbb{C}$. For $z \in \Omega$,

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{\Omega} \frac{\frac{\partial f(\zeta)}{\partial\bar{\zeta}}\, d\zeta \wedge d\bar{\zeta}}{\zeta - z},$$

which is reduced to Cauchy's integral formula when $f(z)$ is holomorphic on U.